B. Math. Hons. IInd year
Second semestral examination 2025
Rings and Modules
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Q 1. Let R be any ring in which the equation $ax = b$ has solutions for any $a \neq 0$ and $b \in R$. Prove:
(i) R has no (left or right) zero-divisors other than 0,
(ii) R has a unity,
(iii) R is a division ring or a field.

OR

Let A be a commutative ring with unity, and let S be a multiplicative subset of A. Prove that $A \setminus S$ is a union of prime ideals of A if, and only if, S is 'saturated' - that is, $st \in S$ implies $s, t \in S$.

Q 2.
(a) For any positive integer $n > 1$, find the number of idempotent elements of the ring $\mathbb{Z}/n\mathbb{Z}$.
(b) If a commutative ring A with unity has exactly 5 ideals, prove that all ideals are principal.

OR

Let $A \subseteq B \subseteq K$ where A is a PID, and K is the quotient field of A. If B is an intermediate subring as above, prove that B must be a PID as well.

Q 3. Let $\theta : \mathbf{C}[X,Y] \to \mathbf{C}[T]$ be the ring homomorphism given by $X \mapsto T^2, Y \mapsto T^{2025}$. Prove that Ker θ is principal.

OR

Let A be a local ring with the maximal ideal $\mathbf{m}$. Let M be a finitely generated A-module and $x_1, \cdots, x_n \in M$ be elements such that $M/\mathbf{m}M$ is generated as an $A/\mathbf{m}$-module by the images of the x_i's. Then prove that M is generated by the x_i's.

Q 4. Let M be a finitely generated module over a PID. Prove that the torsion elements M_{tor} form a submodule, and M/M_{tor} is a free R-module.

OR

Show that each prime number p which is congruent to 1 or 3 mod 8 is expressible as $x^2 + 2y^2$.
Hint. You may assume that -2 is a square mod p for such primes.

Q 5. Let M be the $\mathbb{Z}[i]$-module given as the quotient of the free module $\mathbb{Z}[i]^3$ modulo the relations $f_1 = (1, 3, 6), f_2 = (2 + 3i, -3i, 12 - 18i), f_3 = 2 - 3i, 6 + 9i, -18i)$. Find the cardinality of M.

OR

(a) Let G be the abelian group $\oplus_{i=1}^r \mathbb{Z}/d_i\mathbb{Z}$ where $d_1|d_2| \cdots |d_r$. Prove that the number of endomorphisms of G (group homomorphisms of G to itself) equals $\prod_{i=1}^r |d_i|^{2r-2i+1}$.
(b) Find all the matrices of order 3 up to similarity in $GL_5(\mathbb{Q})$.

OR

Find the rational canonical forms of the two matrices $\begin{pmatrix} 0 & 1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 4 & -3 \end{pmatrix}$, and use this to determine if they are similar over $\mathbb{Q}$.